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A note on the solution of a differential equation arising in boundary-layer theory

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Summary

The differential equation $f''' + ff'' + \lambda f'^2 = 0$ (where dashes denote differentiation with respect to the independent variable η) subject to the boundary conditions $f(0) = 0$, $f'(\infty) = 0$ and either $f'(0) = 1$ or $f''(0) = -1$ is considered. It is shown that by using $p \equiv f'$ as dependent variable and $\phi = C - f$ (where $C = f(\infty)$) as independent variable and then expanding in powers of ϕ, a very good approximation to the solution can be obtained using only a few terms in the expansion.

1. Introduction

The differential equation

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0 \tag{1}$$

subject to boundary conditions that

$$f = 0, \qquad \frac{df}{d\eta} = 1 \quad \text{on} \quad \eta = 0; \qquad \frac{df}{d\eta} \to 0 \quad \text{as} \quad \eta \to \infty, \tag{2}$$

has arisen in several contexts. For example, its solution is required in the problem of a boundary layer on a moving wall (Ackroyd [1]), in magnetohydrodynamic free convection (Singh and Cowling [9] and Riley [7]), in free convective flows in saturated porous media (Cheng [3] and Merkin [5]) and in thermally driven cavity flows in porous media (Blythe et al. [2]). The equation has to be solved numerically. However, we show that it is possible to obtain a solution in the form of a series expansion, relatively few terms of which are required to give good agreement with the exact solution (obtained by numerical integration). In fact the first four terms give a better approximation for $(d^2f/d\eta^2)_0$ than that obtained numerically by Singh and Cowling. Thus we can produce a good approximation for f for all η and have good estimates for starting any numerical integration of the two point boundary value problem given by (1) and (2).

We consider the slightly more general problem

$$\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}+\lambda\left(\frac{df}{d\eta}\right)^2=0 \tag{3}$$

again subject to boundary conditions (2), and where λ is a parameter. From (2), we have that, as $\eta\to\infty, f\to C$ where C is some constant to be determined. The method of solution is to first transform equation (3) into one which has $\phi=C-f$ as independent variable and $p=df/d\eta$ as dependent variable. Then p is expanded as a power series in ϕ; application of the condition that $p=1$ on $\eta=0$ (i.e. $\phi=C$) determines C, with $(d^2f/d\eta^2)_0$ determined from

$$\left(\frac{d^2f}{d\eta^2}\right)_0=(\lambda-1)\int_0^\infty\left(\frac{df}{d\eta}\right)^2 d\eta=(\lambda-1)\int_0^C p\,d\phi \tag{4}$$

which is obtained by an integration of Eqn. (3).

The problem of solving Eqn. (3) subject to the boundary conditions that

$$f=0,\quad \frac{d^2f}{d\eta^2}=-1 \quad\text{on}\quad \eta=0;\qquad \frac{df}{d\eta}\to 0\quad\text{as}\quad \eta\to\infty, \tag{5}$$

can be treated in a similar way. This arises, for example, when a heat flux boundary condition is applied in the problem of free convection in a porous medium and has been solved numerically for $\lambda=-\frac{1}{2}$ by Merkin [6].

2. First problem

Here we are concerned with Eqn. (3) subject to boundary condition (2). Before proceeding to obtain an approximate solution for any λ we note that there are the closed form solutions $f=1-e^{-\eta}$ for $\lambda=-1$ (Riley [8]) and

$$f=\frac{\sqrt{2}\,(1-e^{-\sqrt{2}\eta})}{1+e^{-\sqrt{2}\eta}}$$

for $\lambda=1$ and that when $\lambda=2$ there is no solution of Eqn. (3) which satisfies (2) *. In

* On multiplying by f, Eqn. (3) can be integrated once when $\lambda=2$ to give

$$f\frac{d^2f}{d\eta^2}-\frac{1}{2}\left(\frac{df}{d\eta}\right)^2+f^2\frac{df}{d\eta}=\text{constant}.$$

The constant of integration cannot be chosen to be compatible with the boundary conditions on both $\eta=0$ and as $\eta\to\infty$ as given by (2).

terms of p and ϕ as dependent and independent variables respectively, Eqn. (3) becomes

$$\frac{d}{d\phi}\left(p\frac{dp}{d\phi}\right)+(\phi-C)\frac{dp}{d\phi}+\lambda p=0. \tag{6}$$

Where, as noted above, $p \equiv df/d\eta$, $C=\lim_{\eta\to\infty}f(\eta)$ and $\phi = C-f$. We now expand p in the form

$$p=A_1\phi+A_2\phi^2+A_3\phi^3+A_4\phi^4+A_5\phi^5+\ldots. \tag{7}$$

By substituting expansion (7) into Eqn. (6) and equating like powers of ϕ we can obtain the coefficients $A_1, A_2, \ldots$ in turn. We find that

$$A_1=C, \qquad A_2=-\frac{1+\lambda}{4}, \qquad A_3=\frac{1-\lambda^2}{72C}, \tag{8}$$

$$A_4=\frac{(1-\lambda^2)(1+2\lambda)}{576\,C^2}, \qquad A_5=\frac{(1-\lambda^2)(11+81\lambda+88\lambda^2)}{86400\,C^3}.$$

The condition that $p=1$ when $\phi=C$ then gives

$$C^2\left[1-\frac{1+\lambda}{4}+\frac{1-\lambda^2}{72}+\frac{(1-\lambda^2)(1+2\lambda)}{576}+\frac{(1-\lambda^2)(11+81\lambda+88\lambda^2)}{86400}+\ldots\right]=1. \tag{9}$$

Equation (9) is then an equation which determines C. The skin friction $(d^2f/d\eta^2)_0$ is then determined by using (4) which gives

$$\left(\frac{d^2f}{d\eta^2}\right)_0=C^3(\lambda-1)\left[\frac{1}{2}-\frac{1+\lambda}{12}+\frac{1-\lambda^2}{288}+\frac{(1-\lambda^2)(1+2\lambda)}{2880}\right.$$
$$\left.+\frac{(1-\lambda^2)(11+81\lambda+88\lambda^2)}{518400}+\ldots\right]. \tag{10}$$

We can see that the terms in the series (9) and (10) rapidly decrease, suggesting that good estimates for C and the skin friction can be obtained with relatively few terms. This can be seen from Table 1 where, for $\lambda=0$, values of C and $(d^2f/d\eta^2)_0$ are given, being obtained by taking successively more terms in the series (9) and (10). It can be seen that these rapidly approach their computed values of 1.14277 and -0.62755 respectively. This is in contrast with the method employed by Ackroyd [1] who obtained an expansion valid for large η, requiring 17 terms to obtain an equivalent accuracy. Also, as can be seen from Table 1, the error in the series solution changes sign between the fourth and fifth approximations, suggesting that the series (7) could well be asymptotic. This point has been examined by considering the coefficient of the term of $O(\phi^6)$ in (7) for $\lambda=0$. This is found to be $-(115200C^4)^{-1}$, which is smaller than A_5 and different in sign to it. When this extra term is used in (9) and (10), these give 1.14277 and -0.62755 (the numerically computed values) for C and $f''(0)$ respectively.

Table 1. Values of C and $(d^2f/d\eta^2)_0$ for $\lambda = 0$ obtained by increasing the number of terms taken in series (9) and (10)

Number of terms	C	$\left(\frac{d^2f}{d\eta^2}\right)_0$
1	1.00000	−0.50000
2	1.15470	−0.64150
3	1.14416	−0.62929
4	1.14286	−0.62766
5	1.14276	−0.62754

Values of C and $(d^2f/d\eta^2)_0$ obtained from (9) and (10) for various values of λ are given in Table 2 as well as their values obtained by solving the two-point boundary-value problem numerically. This has to be done iteratively, reasonable estimates for the unknown boundary conditions being needed for the process to converge. These were supplied by their approximate values obtained as above. We notice that the series method gives the correct value for $\lambda = 1$ and $\lambda = -1$, and that the agreement with the numerically computed results is good for most of the values of λ considered; for $\lambda = -2$ the errors in C and $(d^2f/d\eta^2)_0$ being 0.1% and 0.2% respectively. This error decreases as λ increases from $\lambda = -2$. However, as we approach the case $\lambda = 2$ the difference becomes greater, at $\lambda = 1.8$ the errors in C and $(d^2f/d\eta^2)_0$ are 7% and 19% respectively. This is to be expected

Table 2. Values of C and $(d^2f/d\eta^2)_0$ for various λ obtained from series (9) and (10) and obtained numerically

λ	Series		Numerically	
	C	$\left(\frac{d^2f}{d\eta^2}\right)_0$	C	$\left(\frac{d^2f}{d\eta^2}\right)_0$
−2.0	0.90648	−1.28461	0.90563	−1.28181
−1.8	0.92249	−1.23150	0.92204	−1.23009
−1.6	0.93974	−1.17693	0.93957	−1.17632
−1.4	0.95832	−1.12046	0.95824	−1.12026
−1.2	0.97835	−1.06164	0.97833	−1.06160
−1.0	1.00000	−1.00000	1.00000	−1.00000
−0.8	1.02348	−0.93502	1.02348	−0.93501
−0.6	1.04908	−0.86610	1.04908	−0.86609
−0.4	1.07715	−0.79253	1.07715	−0.79254
−0.2	1.10816	−0.71341	1.10817	−0.71343
0	1.14276	−0.62754	1.14277	−0.62755
0.2	1.18179	−0.53326	1.18175	−0.53323
0.4	1.22642	−0.42818	1.22629	−0.42808
0.6	1.27827	−0.30869	1.27800	−0.30853
0.8	1.33969	−0.16911	1.33933	−0.16900
1.0	1.41421	0.00000	1.41421	0.00000
1.2	1.50739	0.21539	1.50945	0.21616
1.4	1.62858	0.50944	1.63839	0.51774
1.6	1.79503	0.95274	1.83302	1.00913
1.8	2.04319	1.72912	2.20605	2.14634

as the numerical solutions strongly indicate that the case $\lambda = 2$ (where there is no solution of (2) and (3)) is approached in a singular way.

From Table 1 we can see that the first two terms in series (9) give a good estimate for C (being about 1% in error). This suggests that we can obtain an approximate expression for $f(\eta)$ from the first two terms in (7) by a simple integration. On doing this we find that

$$C = \frac{2}{\sqrt{3-\lambda}} \quad \text{and} \quad f(\eta) = \frac{C(1-\mathrm{e}^{-C\eta})}{1+(C^2-1)\,\mathrm{e}^{-C\eta}}. \tag{11}$$

Equations (11) give the correct solution when $\lambda = 1$ and $\lambda = -1$. On comparing (11) with the numerically calculated values we find, for $\lambda = 0$, that (11) is in error by at most 1%.

It is interesting to contrast the method of solution given above with that used by Meksyn [4] on similar problems. The advantage of the present method is that C^2 factors out of the terms in the series in (9) and consequently only a simple calculation is required to obtain C (with $f''(0)$ evaluated in a similarly straightforward way). However, to obtain successfully better approximations, Meksyn's method requires the solution of successively higher order algebraic equations, thus rendering his technique much less useful for the sort of problem given by Eqn. (3) with boundary conditions (2) or (5).

3. Second problem

Here we are concerned with the solution of Eqn. (3) subject to boundary conditions (5). We notice that there are special cases; when $\lambda = -1$, $f = 1 - \mathrm{e}^{-\eta}$ and when $\lambda = 1$ there is no solution which will satisfy (5). Again we start from Eqn. (6) with the coefficients in expansion (7) still being given by (8). The difference is in the application of the boundary conditions. Using (4) we have now that

$$(1-\lambda)\int_0^C p\,\mathrm{d}\phi = 1 \tag{12}$$

which we use to determine C. This gives

$$(1-\lambda)C^3\left[\frac{1}{2} - \frac{1+\lambda}{12} + \frac{1-\lambda^2}{288} + \frac{(1-\lambda^2)(1+2\lambda)}{2880} + \frac{(1-\lambda^2)(11+81\lambda+88\lambda^2)}{518400} + \dots\right] = 1. \tag{13}$$

Equation (13) confirms that there is no solution when $\lambda = 1$. The value of $(\mathrm{d}f/\mathrm{d}\eta)_0$ is then found from

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)_0 = C^2\left[1 - \frac{1+\lambda}{4} + \frac{1-\lambda^2}{72} + \frac{(1-\lambda^2)(1+2\lambda)}{576} + \frac{(1-\lambda^2)(11+81\lambda+88\lambda^2)}{86400} + \dots\right]. \tag{14}$$

Table 3. Values of C and $(df/d\eta)_0$ for various λ as obtained from (13) and (14) and obtained numerically

λ	Series		Numerically	
	C	$\left(\frac{df}{d\eta}\right)_0$	C	$\left(\frac{df}{d\eta}\right)_0$
−2.0	0.83388	0.84622	0.83369	0.84746
−1.8	0.86063	0.87038	0.86053	0.87105
−1.6	0.89007	0.89708	0.89002	0.89739
−1.4	0.92266	0.92698	0.92264	0.92709
−1.2	0.95903	0.96091	0.95902	0.96093
−1.0	1.00000	1.00000	1.00000	1.00000
−0.8	1.04667	1.04581	1.04666	1.04581
−0.6	1.10057	1.10058	1.10057	1.10058
−0.4	1.16400	1.16768	1.16396	1.16767
−0.2	1.24020	1.25249	1.24020	1.25247
0	1.33478	1.36429	1.33478	1.36427
0.2	1.45734	1.52069	1.45733	1.52074
0.4	1.62715	1.76028	1.62713	1.76056
0.6	1.89138	2.18935	1.89132	2.19012
0.8	2.42259	3.27003	2.42250	3.27151
0.9	3.07906	5.01432	3.07897	5.01608

Values of C and $(df/d\eta)_0$ determined from (13) and (14) and obtained by a numerical solution of the problem are given in Table 3. Again we can see that there is very good agreement between the approximate values and the numerically computed values, with the largest percentage error occurring at $\lambda = -2$ (C and $(df/d\eta)_0$ being in error by about 0.02% and 0.15% respectively there). The results agree very well even close to the critical case $\lambda = 1$, and over much of the range of λ considered the difference between the series solution and the computed values is extremely small. The method appears to give better agreement for this problem than for the first one considered. This is perhaps to be expected as the terms in series (13) (for the determination of C) decrease more rapidly than those in (9). In fact the first two terms in (13) and (14) give values for C and $(df/d\eta)_0$ for $\lambda = 0$ as 1.33887 and 1.34442, being in error by 0.3% and 1.5% respectively.

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